Second-order Synchrosqueezing for Compact Invertible Time-Frequency and Time-Scale Representations

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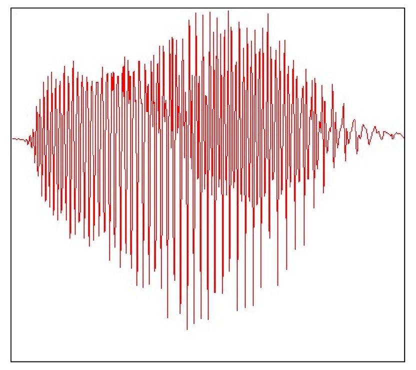
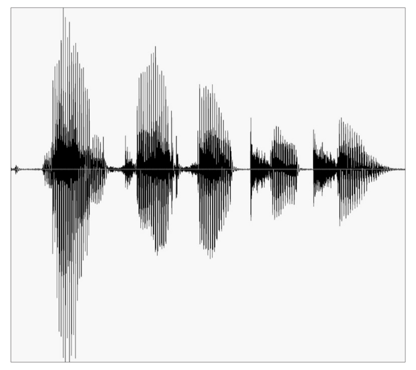
Introduction: multicomponent signals (MCS)

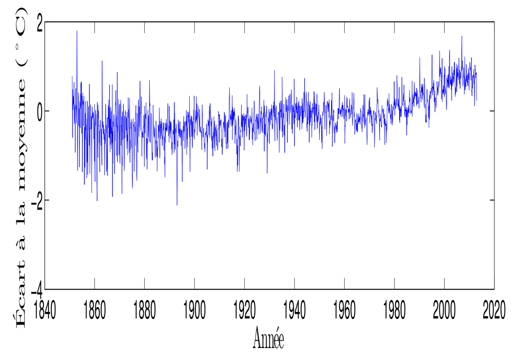
- Analyzing and representing **multicomponent signals**, ie superpositions of AM–FM modes:

$$s(t) = \sum_{k=1}^{K} s_k(t) = \sum_{k=1}^{K} A_k(t)\cos(2\pi\phi_k(t))$$

- Multicomponent signals in real life:







- Two distinct problems
  - The separation of the modes: get s_k from signal s
  - The demodulation: estimate the A_k and ϕ'_k
- A solution: invertible ideal time-frequency or time-scale representations

$$\mathrm{TI}_s(\eta, t) = A_k(t)\delta(\eta - \phi'_k(t)).$$

Linear time-frequency (TF) and time-scale (TS) transforms

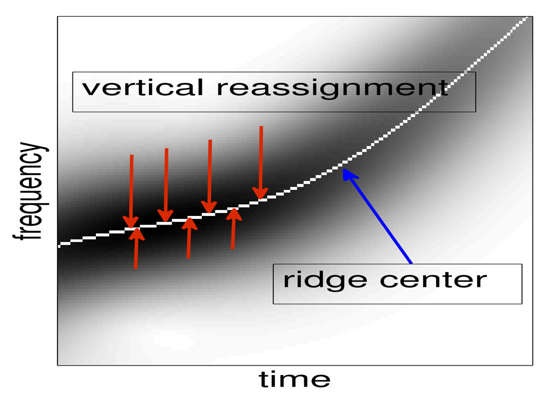
- Local versions of the Fourier transform of a signal $f \in L^1(\mathbb{R})$: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t)\, e^{-2i\pi\xi t}\, dt$
- Using analytic wavelets and windows allows us to consider equally real or complex MCS of the type:

$$s(t) = \sum_{k=1}^{K} s_k(t) = \sum_{k=1}^{K} A_k(t) e^{2i\pi\phi_k(t)}$$

Short-time Fourier transform (STFT)	Continuous wavelet transform (CWT)
Depends on a window g : $V_f^g(\eta, t) = \langle f, g_{\eta,t}\rangle = \int_{\mathbb{R}} f(\tau)g(\tau - t)e^{-2i\pi\eta(\tau - t)}\, d\tau$	Depends on an admissible wavelet ψ (ie, $0 < C_\psi = \int_0^\infty \lvert\hat{\psi}(\eta)\rvert^2 \frac{d\eta}{\eta} < \infty$): $W_f(a, t) = \langle f, \psi_{a,t}\rangle = \frac{1}{a}\int_{\mathbb{R}} f(\tau)\psi\left(\frac{\tau - t}{a}\right)^* d\tau$
Pointwise reconstruction: $f(t) = \frac{1}{g(0)}\int_{\mathbb{R}} V_f(\eta, t)d\eta$	Morlet formula: $f(t) = \frac{1}{C'_\psi}\int_0^\infty W_f(a, t)\frac{da}{a}$, where $C'_\psi = \int_{\mathbb{R}} \frac{\hat{\psi}(\eta)^*}{\eta}\, d\eta$.
First order approximation of MCS: $V_s(\eta, t) \approx \sum_k s_k(t)\hat{g}(\eta - \phi'_k(t))$ Low modulation assumptions: $\lvert A'_k(t)\rvert, \lvert\phi''_k(t)\rvert \ll 1$ Separation assumption: $\phi'_{k+1} - \phi'_k > 2\Delta$ where $\mathrm{supp}\,\hat{g} \subset [-\Delta, \Delta]$	First order approximation of MCS: $W_s(a, t) \approx \sum_k s_k(t)\hat{\psi}(a\phi'_k(t))^*$. Low modulation assumptions: $\lvert A'_k(t)\rvert, \lvert\phi''_k(t)\rvert \ll \phi'_k(t)$ Separation assumption: $\frac{\phi'_{k+1} - \phi'_k}{\phi'_{k+1} + \phi'_k} > \Delta$ where $\mathrm{supp}\,\hat{\psi} \subset [1 - \Delta, 1 + \Delta]$

Reassignment methods: sharpening the representations

- based on reassignment operators:
  - Local instantaneous frequency $\hat{\omega}_f(\eta, t) = \frac{1}{2\pi}\partial_t \arg V_f(\eta, t)$
  - Local group delay $\hat{\tau}_f(\eta, t) = t - \frac{1}{2\pi}\partial_\eta \arg V_f(\eta, t)$
- Mapping of the coefficients:
  - vertical $(\eta, t) \mapsto (\hat{\omega}_f, t)$ for the Synchrosqueezing (SST)
  - oblique $(\eta, t) \mapsto (\hat{\omega}_f, \hat{\tau}_f)$ for the Reassignment (RM)



SST vs RM: reconstruction or representation

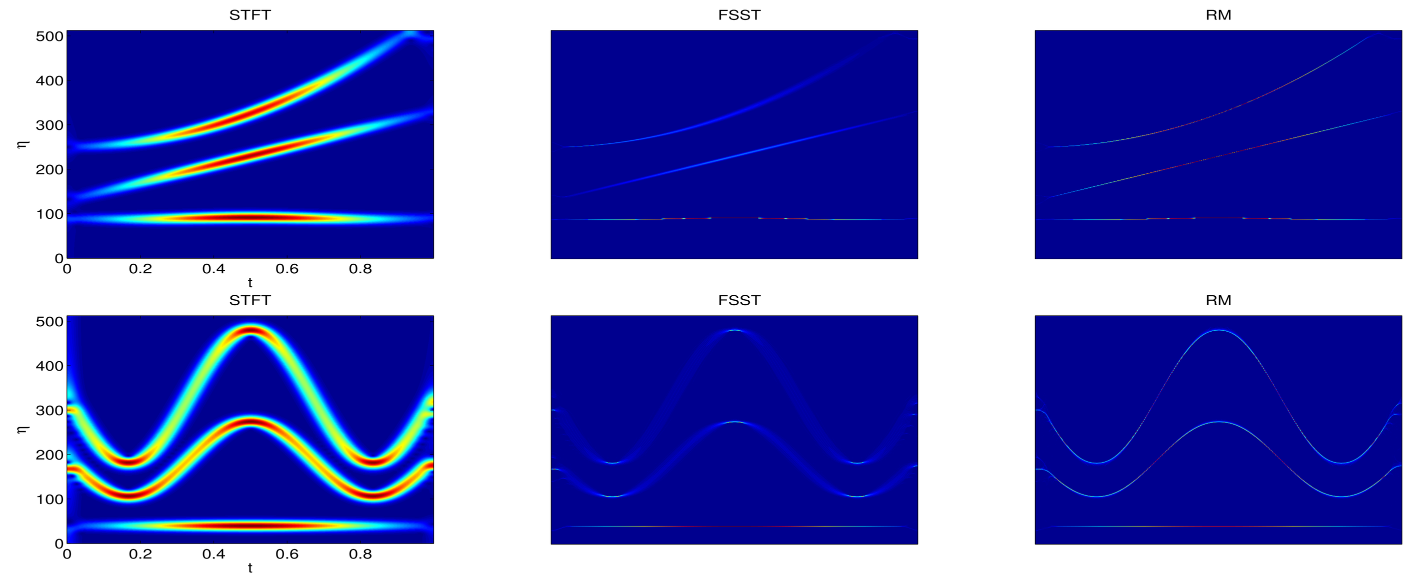
- SST:

$$T_f(\omega, t) = \frac{1}{g(0)}\int_0^\infty V_f(\eta, t)\delta\left(\omega - \hat{\omega}_f(\eta, t)\right)\, d\eta.$$

  - Ideal representation of pure waves (ϕ_k linear)
  - Reconstruction
- RM:

$$R_f(\omega, \tau) = \int_{\mathbb{R}}\int_0^\infty \lvert V_f(\eta, t)\rvert^2 \delta(\omega - \hat{\omega}_f(\eta, t))\delta(\tau - \hat{\tau}_f(\eta, t))\, d\eta dt.$$

  - Ideal representation of linear chirps (ϕ_k quadratic)
  - No reconstruction



Toward second-order SST's

Estimating the frequency modulation $\phi''_k(t)$:

$$\hat{q}_f(\eta, t) = \frac{\partial_t\hat{\omega}_f(\eta, t)}{\partial_t\hat{\tau}_f(\eta, t)}.$$

Exact estimation for a linear chirp

Easily computed:

$$\hat{q}_f(\eta, t) = \mathcal{I}m\left\{\frac{1}{2\pi}\frac{(V_f^{g'})^2 - V_f^{g''}V_f^g}{V_f^{xg}V_f^{g'} - V_f^{xg'}V_f^g}\right\},$$

($V_f^{g''}$ and $V_f^{xg'}$: windows g'' and $x \mapsto xg'(x)$)

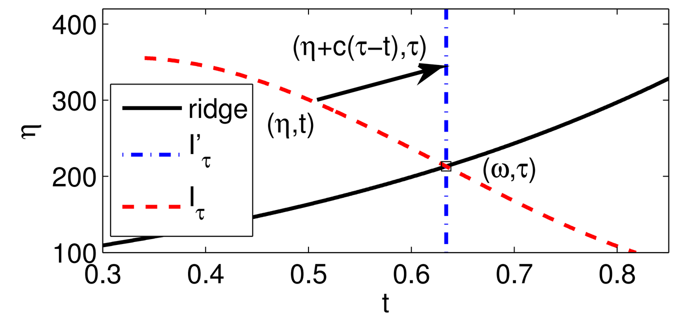
Vertical SST (VSST): Taking into account second-order terms in standard SST

$$\tilde{\omega}_f(\eta, t) = \begin{cases} \hat{\omega}_f(\eta, t) + \hat{q}_f(\eta, t)(t - \hat{\tau}_f(\eta, t)) & \text{if } \partial_t\hat{\tau}_f(\eta, t) \neq 0 \\ \hat{\omega}_f(\eta, t) & \text{otherwise.} \end{cases}$$

The vertical second-order synchrosqueezing (VSST):

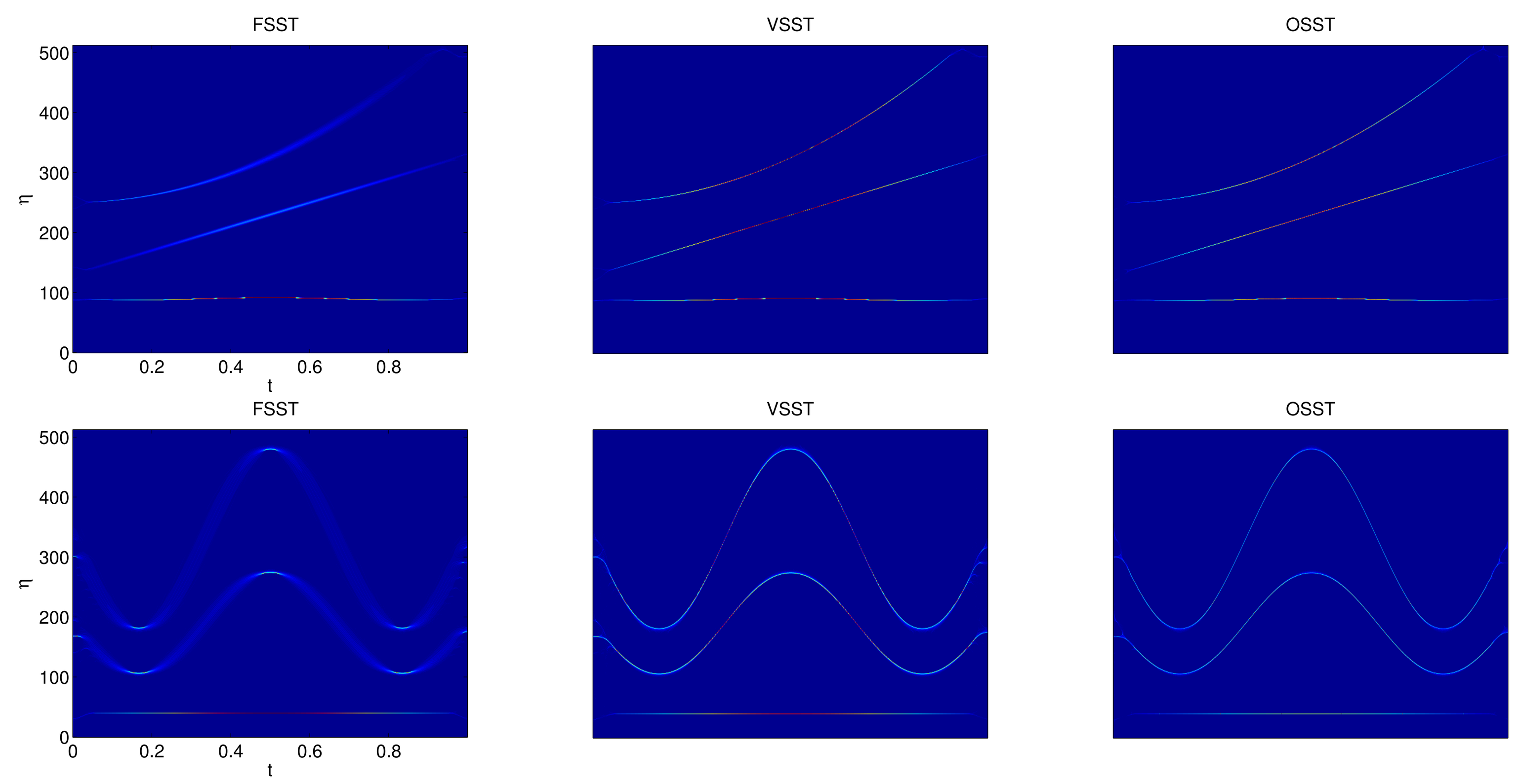
$$TV_f(\eta, t) = \frac{1}{g(0)}\int_{\mathbb{R}} V_f(\eta, t)\delta\left(\omega - \tilde{\omega}_f(\eta, t)\right)\, d\eta$$

Oblique SST (OSST): Reassignment with complex, phase-corrected coefficients

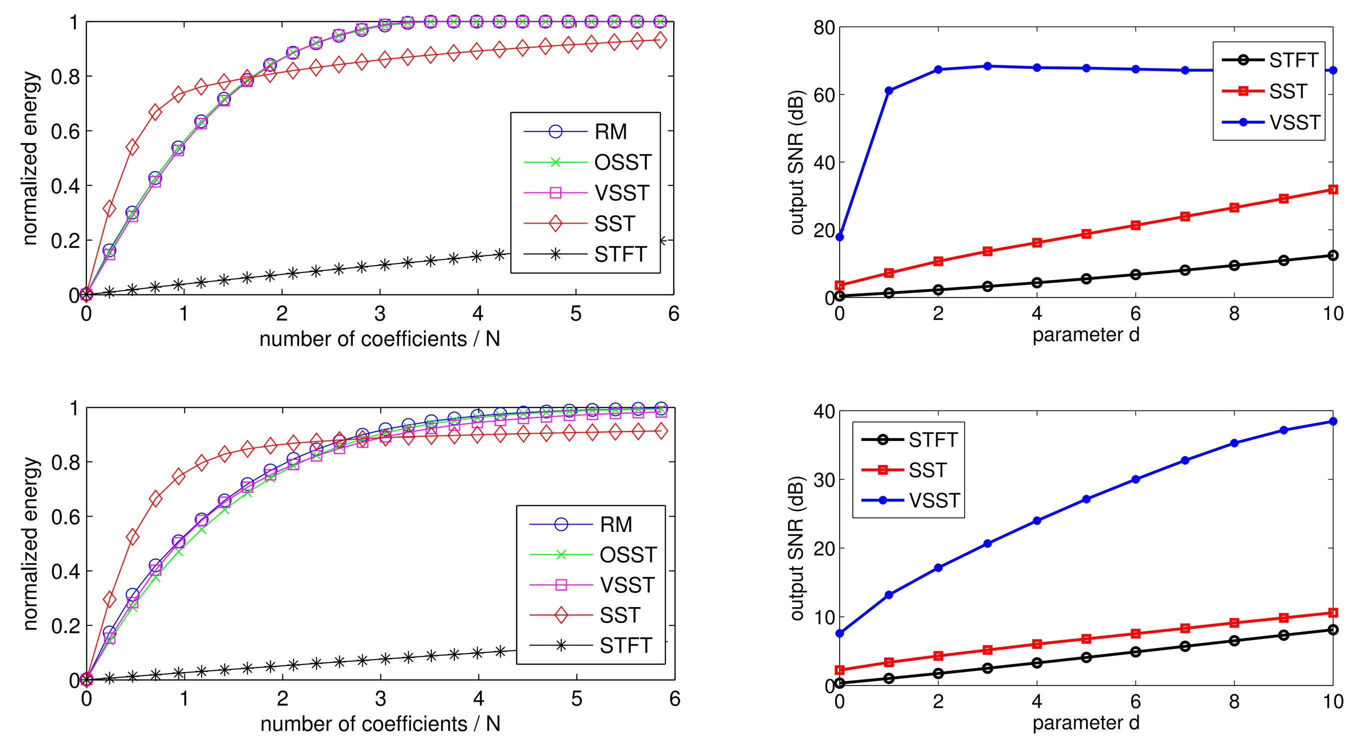


$$TO_f(\omega, \tau) = \frac{1}{g(0)}\iint_{\mathbb{R}^2} V_f(\eta, t)e^{i\pi(2\omega - \hat{q}_f(\eta,t)(\tau - t))(\tau - t)}\delta\left(\omega - \hat{\omega}_f(\eta, t)\right)\delta\left(\tau - \hat{\tau}_f(\eta, t)\right)\, d\eta dt.$$

Numerical results



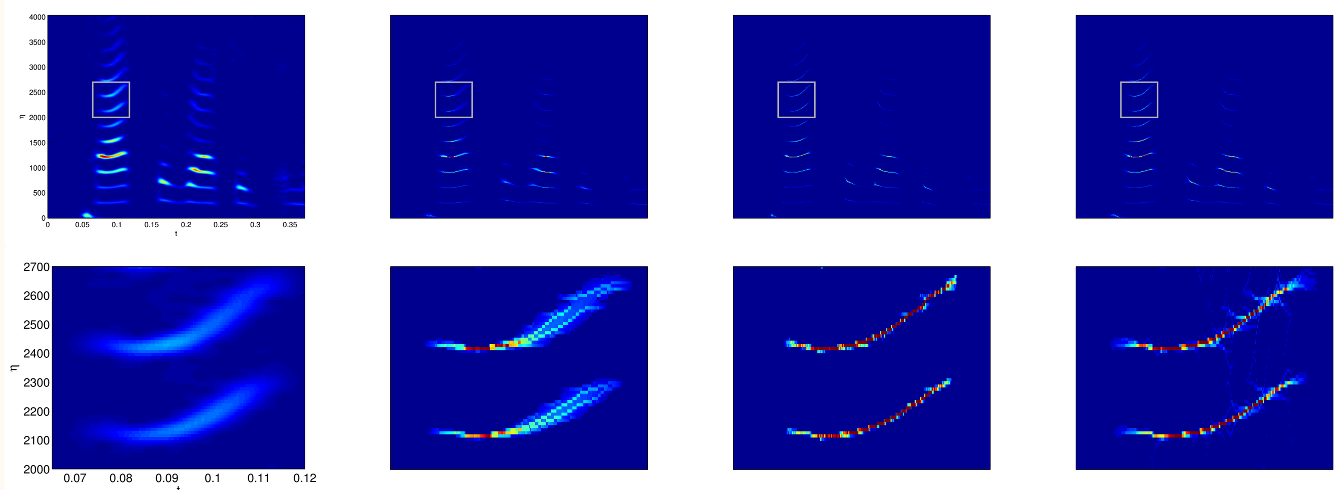
SST OSST VSST



Normalized energy Nonlinear approximation

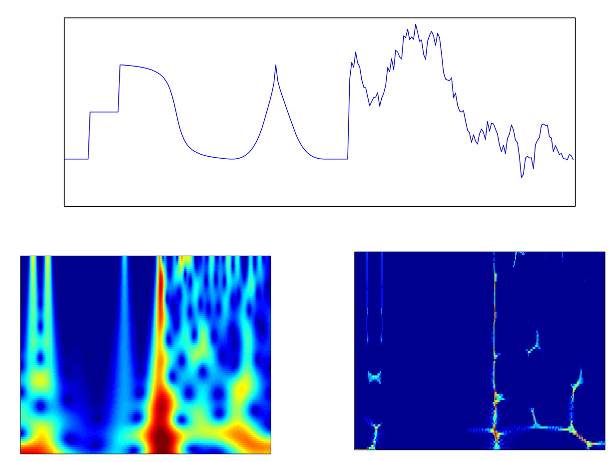
Perspectives, potential applications

Second-order SST of a speech signal



STFT SST OSST VSST

Second-order SST for the CWT



CWT Reassigned CWT

Future works

- Improving the ridge detection: using tools designed for reassigned transforms
- Application for real-life signals (detection, monitoring, etc)
- Extension to random signals
- Extension in dimension 2 (with monogenic SST)
- Sparsification of discrete representations (Gabor frames, etc)

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