Synchrosqueezing transform, reassignment and ridge analysis

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Introduction

- Time-frequency, time-scale analysis of multicomponent signals.
- Continuous Wavelet transform of chirps: Ridge analysis.
- The SynchroSqueezed transform (SST) [1] has two purposes:
  - sharpening the time-frequency representation from the wavelet (or the windowed Fourier) transform;
  - reconstructing the modes automatically.
- Here: a presentation of the SST, an insight into its good mathematical properties. Then, an illustration of the transform with two examples, and an overview of some alternatives and remaining perspectives.

The continuous wavelet transform

Fourier transform of a signal $f \in L^1(\mathbb{R})$: $\hat{f}(\xi) = \int_{\mathbb{R}} f(t)\, e^{-2i\pi\xi t}\, dt$.

Given an admissible wavelet $\psi \in \mathcal{S}(\mathbb{R})$, the continuous wavelet transform of f:

$$W_f(a,b) = \left\langle f, \psi_{a,b} \right\rangle = \frac{1}{a}\int_{\mathbb{R}} f(t)\, \overline{\psi\left(\frac{t-b}{a}\right)}\, dt.$$

Assumption on the mother wavelet:

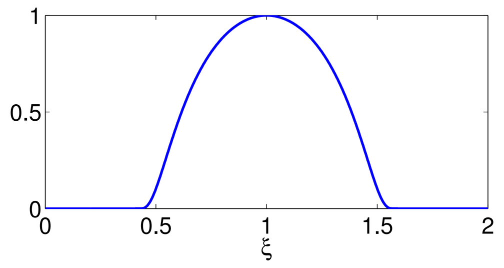
- $\hat{\psi}$ is centered in 1:

$$1 = \arg\max_{\xi} |\hat{\psi}(\xi)|$$

- $\hat{\psi}$ is compactly supported in $[1-\Delta, 1+\Delta]$.

Example : the bump wavelet:

$$\hat{\psi}_{Bump}(\xi) = e^{1 - \frac{1}{1-\left(\frac{\xi-1}{\Delta}\right)^2}} \chi_{[1-\Delta,1+\Delta]}$$



$\hat{\psi}_{Bump}(\xi)$ with $\Delta = 0.6$

Multicomponent signals in the time-scale plane

Multicomponent signals

- Mode (or component): a signal $a(t)e^{2i\pi\phi(t)}$ with $a(t) > 0$, $\phi'(t) > 0$ and $|a'(t)|, |\phi''(t)| \ll \phi'(t)$.
- Multicomponent signal f: a sum of modes: $f(t) = \sum_{k=1}^{K} f_k(t) = \sum_{k=1}^{K} a_k(t) e^{2i\pi\phi_k(t)}$

Corresponding wavelet transform $W_f(a,b) = \sum_{k=1}^{K} a_k(b) e^{2i\pi\phi_k(b)} \overline{\hat{\psi}(a\phi_k'(b))} + O\left(\frac{|a_k'(t)|}{|\phi_k'(t)|}, \frac{|\phi_k''(t)|}{|\phi_k'(t)|}\right)$.

The wavelet transform of a multicomponent signal forms several ridges, centered at the scale $a = \frac{1}{\phi_k'(b)}$.

Separation condition: If $\phi_{k+1}' \gg \phi_k' \gg \cdots$, the modes are separated, each of them being distinctly visible on the spectrogram.

Real signal and analytic wavelets: In practice one often deals with real chirps $f_k(t) = a_k(t)\cos(2\pi\phi_k(t))$. If we assume the wavelet ψ to be analytic (i.e. $\hat{\psi}(\xi) = 0$ if $\xi < 0$), then the above approximation remains approximately true up to a factor $\frac{1}{2}$.

Synchrosqueezing spirit

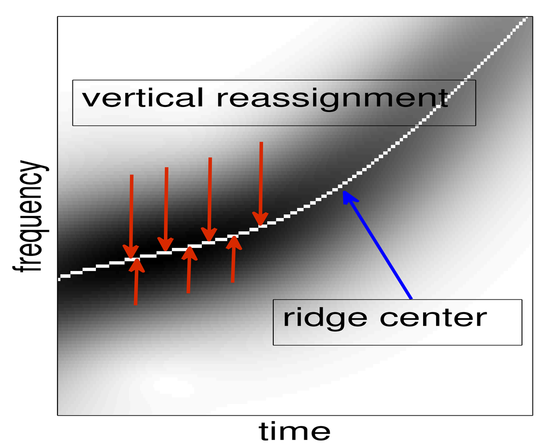
A kind of Reassignment

- Compute the candidate instantaneous frequency

$$\omega_f(a,b) = \mathcal{R}e\left\{\frac{\partial_b W_f(a,b)}{2i\pi W_f(a,b)}\right\}. \quad (1)$$

- Near the ridge of mode k, $\omega_f(a,b) \approx \phi_k'(b)$.
- Reassigning the WT: $(a,b) \mapsto (\omega_f(a,b), b)$
- Using the (truncated) Morlet synthesis formula:

$$T_f(\xi,b) = \int_{\{|W_f(a,b)|>\varepsilon\}\cap\{\omega_f(a,b)\approx\xi\}} W_f(a,b)\frac{da}{a}. \quad (2)$$



We expect to get $T_f(\phi_k'(b), b) \approx f_k(b)$.

Link with classical reassignment: If one defines the group delay $\tau_f(a,b) = b - \mathcal{R}e\left\{\frac{a\partial_a(aW_f(a,b))}{2i\pi W_f(a,b)}\right\}$, one can *reassign* the scalogram $|W_f(a,b)|^2$ acording to the map $(a,b) \mapsto (\omega_f(a,b), \tau_f(a,b))$. This enables to take into account stronger frequency modulations, but no reconstruction is available

Mathematical study

Assumptions on the modes

The modes $f_k(t) = a_k(t)e^{2i\pi\phi_k(t)}$ satisfies

$$a_k \in \mathcal{C}^1(\mathbb{R}) \cap L_\infty(\mathbb{R}) \text{ and } \phi_k \in \mathcal{C}^2(\mathbb{R}).$$
$$\inf_t \phi_k'(t) > 0 \text{ and } \sup_t \phi_k'(t) < \infty,$$
$$|a_k'(t)|, |\phi_k''(t)| \le \varepsilon|\phi_k'(t)|,$$
$$M = \sup_t \phi''(t) < \infty.$$

Moreover the modes are separated, i.e. $\forall k$,

$$\left|\frac{\phi_{k+1}'(t) - \phi_k'(t)}{\phi_{k+1}'(t) + \phi_k'(t)}\right| > \frac{1}{1-\Delta}.$$

Uniform approximation

In the vicinity of time b one can approximate $f_k(t)$ by the pure wave $\tilde{f}_k^b(t) = a_k(b)e^{2i\pi[\phi_k(b)+\phi_k'(b)(t-b)]}$ with good accuracy. Wavelet coefficient $W_{f_k}(a,b)$ will be approximated by

$$W_{\tilde{f}_k^b}(a,b) = f_k(b)\overline{\hat{\psi}}(a\phi_k'(b)).$$

Moreover, W_f is concentrated in the union of strips $Z_k = \{(a,b); |a\phi_k'(b) - 1| < \Delta\}$. The key idea is then to show that the error $|W_f - W_{\tilde{f}^b}|$ is uniformly bounded on this compact set.

Theorem of Daubechies et al.

Given a unit window $h \in \mathcal{S}(\mathbb{R})$, a threshold $\tilde{\varepsilon}$ and a resolution $\delta > 0$, one defines the SST of f by

$$T_{f,\tilde{\varepsilon}}^{h,\delta}(\xi,b) = \int_{\{a;|W_f(a,b)|>\tilde{\varepsilon}\}} W_f(a,b)\frac{1}{\delta}h\left(\frac{\xi - \omega_f(a,b)}{\delta}\right)\frac{da}{a}. \quad (3)$$

The theorem in [1] shows that, provided $\tilde{\varepsilon}$ is small enough,

- $|W_f(a,b)| > \tilde{\varepsilon}$ only when $(a,b) \in \cup Z_k$.
- $\forall (a,b) \in Z_k$, $|\omega(a,b) - \phi_k'(b)| \le \tilde{\varepsilon}$.
- One can reconstruct $f_k(b)$ with accuracy $\tilde{\varepsilon}$ by summing $T_f(\xi,b)$ around $\xi = \phi_k'(b)$.

However, in practice one needs to know the value $\phi_k'(b)$, which can be hard to estimate.

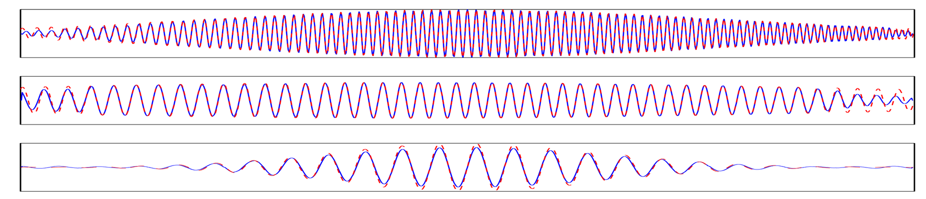
Illustrations

A toy example

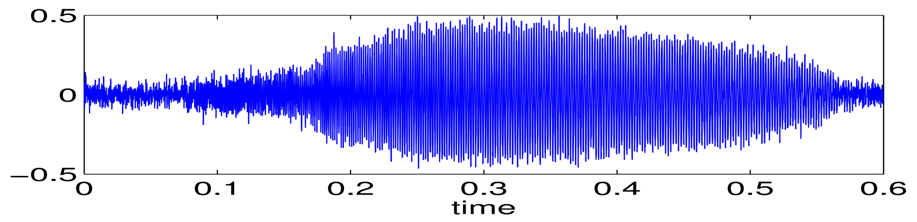

Wavelet transform $W_f(a,b)$


Frequency information $\omega_f(a,b)$

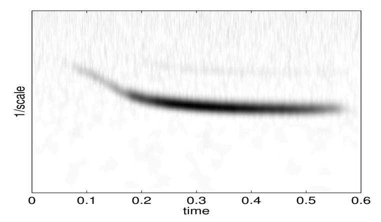

SST = WT after a mapping $a \mapsto \omega_f(a,b)$.

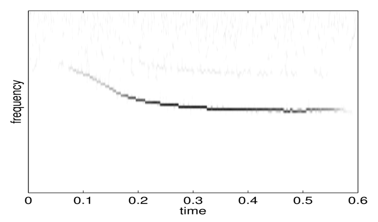

The 3 corresponding modes (plain, blue) and the true components (dashed, red).

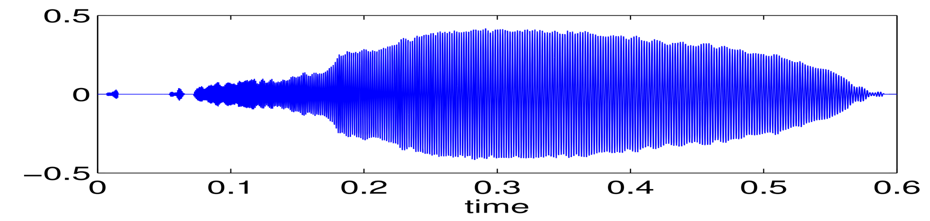
Denoising a bat sound



Original noisy bat signal


Wavelet transform W_f


SST T_f

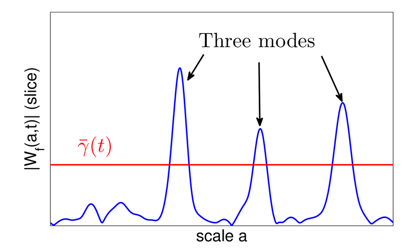


Reconstruction of the first component

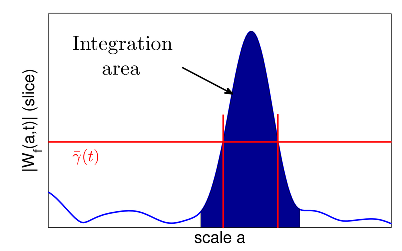
Some variants, alternatives and perspectives

Direct reconstruction from a local Morlet formula

The idea of using an integration in scales (i.e., the Morlet formula) for mode reconstruction can be applied directly from the wavelet transform [2]; this can provide some benfits compared to the classical ridge reconstruction. As usual, this is done directly from the wavelet transform by a detection step, and a reconstruction.



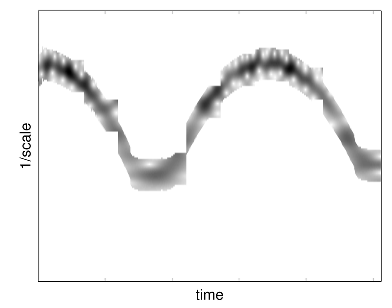
Slice of the wavelet transform of a 3-component signal

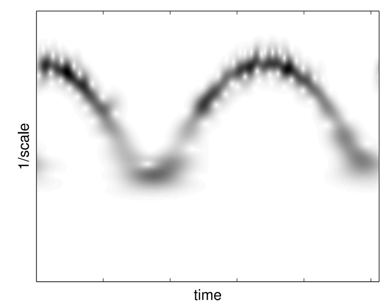


Integration area for reconstruction

Smoothing the modes using a RKHS projection

Problem in reconstruction when the ridge is not correctly detected: in this case the Morlet formula is no longer valid. [3] proposes a regularization based on RKHS projection. This ensures in particular to reconstruct a smooth mode.


A truncated wavelet transform, where the ridge has not been perfectly estimated.


The same wavelet transform after RKHS regularization.

Current and future works

- Taking into account non-negligible frequency modulations
- Improving the ridge detection: using tools designed for reassigned transforms
- Comparing with classical ridge analysis, or with EMD
- Extending the SST in 2 or 3 dimensions: the monogenic SST.

Bibliography

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[2] S. Meignen, T. Oberlin, and S. McLaughlin,
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[3] T. Oberlin, S. Meignen, and V. Perrier,
"On the mode synthesis in the synchrosqueezing method,"
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Conclusion

- The SST: a powerful tool for multicomponent signal analysis and processing.
- Combines a sharp and convenient representation (like reassignment) and a reconstruction (like classical ridge analysis).
- Comes with powerful mathematical statements.
- Offers a lot of interesting perspectives (theory, extensions, applications).